

# Definition of fractal measures arising from fractional calculus

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The sets and curves of fractional dimension have been constructed and found to be useful at number of places in science [1]. They are used to model various irregular phenomena. It is wellknown that the usual calculus is inadequate to handle such structures and processes. Therefore a new calculus should be developed which incorporates fractals naturally. Fractional calculus, which is a branch of mathematics dealing with derivatives and integrals of fractional order, is one such candidate. The relation between ordinary calculus and measures on  $\mathbb{R}^n$  is wellknown. For example, an  $n$ -fold integration gives an  $n$ -dimensional volume. Also, the solution of  $df/dx = 1_{[0,x]}$ , where  $1_{[0,x]}$  is an indicator function of  $[0, x]$ , gives length of the interval  $[0, x]$  [2]. The aim of this paper is to arrive at a definition of a fractal measure using the concepts from the fractional calculus. Here we shall restrict ourselves to simple subsets of  $[0, 1]$  and more rigorous treatment will be given elsewhere.

We first define a differential of fractional order  $\alpha$  ( $0 \leq \alpha \leq 1$ ) as follows:  $d^\alpha x = d^{-\alpha} 1_{dx}(x)/dx^{-\alpha}$  where

$$\frac{d^q f(x)}{[d(x-a)]^q} = \frac{1}{\Gamma(-q)} \int_a^x \frac{f(y)}{(x-y)^{q+1}} dy, \quad \text{for } q < 0, \quad (1)$$

is the Riemann-Liouville fractional integral [3]. Now we define a "fractal integral" by  ${}_a\mathbb{D}_b^{-\alpha} f(x) = \int_a^b f(y) d^\alpha y$ , written in discrete form as,  ${}_a\mathbb{D}_b^{-\alpha} f(x) = \lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} f(x_i^*) d^{-\alpha} 1_{dx_i} / [d(x_{i+1} - x_i)]^{-\alpha}$ , where  $[x_i, x_{i+1}]$ ,  $i = 0, \dots, N-1$ ,  $x_0 = a$  and  $x_N = b$ , provide a partition of the interval  $[a, b]$  and  $x_i^*$  is some suitably chosen point of the subinterval  $[x_i, x_{i+1}]$ . We now define the fractional measure of a subset  $A \cap [0, x]$  (assuming it to be measurable) as  $\mathcal{F}^\alpha(A \cap [0, x]) = {}_0\mathbb{D}_x^{-\alpha} 1_A(x)$ . Consider an example of a one-third Cantor set  $C$  with dimension  $d = \ln(2)/\ln(3)$ . For this set  $\mathcal{F}$  can be written as  $\mathcal{F}^\alpha(C) = {}_0\mathbb{D}_1^{-\alpha} 1_C(x)$ . Now we choose  $x_i^*$  to be such that  $1_C(x_i^*)$  is the maximum in that interval, then

$$\mathcal{F}^\alpha(C) = \lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} F_C^i \frac{(x_{i+1} - x_i)^\alpha}{\Gamma(\alpha + 1)}, \quad (2)$$

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where  $F_C^i$  is a flag function which is 1 if a point of set  $C$  belongs to the interval  $[x_i, x_{i+1}]$  and zero otherwise. Clearly this measure is infinite if  $\alpha < d$  and zero if  $\alpha > d$ . At  $\alpha = d$  we have  $\mathcal{F}^d(C) = 1/\Gamma(d+1)$  whereas the Hausdorff measure [1]  $\mathcal{H}^d(C) = [\Gamma(1/2)]^d/\Gamma(1+d/2)$ .

Recently, a new quantity viz. local fractional derivative (LFD), was defined [4] as

$$\mathcal{ID}^q f(y) = \lim_{x \rightarrow y} \frac{d^q[f(x) - f(y)]}{[d(x-y)]^q} \quad 0 < q \leq 1, \quad x > y, \quad (3)$$

where the RHS uses Riemann-Liouville fractional derivative [3] given by

$$\frac{d^q f(x)}{[d(x-a)]^q} = \frac{1}{\Gamma(1-q)} \frac{d}{dx} \int_a^x \frac{f(y)}{(x-y)^q} dy \quad \text{for } 0 < q < 1. \quad (4)$$

We also introduced [5] local fractional differential equations which involve LFDs. A solution of the equation [5]  $\mathcal{ID}^\alpha f(x) = 1_C(x)$  turns out to be equation (2) implying  $f(x) = \mathcal{F}^\alpha(C \cap [0, x])$  [2]. This generalizes the fact that the solution of  $f'(x) = 1_{[0,x]}$  is the length of the interval  $[0, x]$ .

A local fractional diffusion equation given by  $\mathcal{ID}_t^\alpha W(x, t) = (1_C(t)/2)(\partial^2 W(x, t)/\partial x^2)$  (compare Ref. [5]), where  $W(x, t)$  is a probability density for finding a particle in neighbourhood of  $x$  at time  $t$ , has a solution given by [5], for  $W(x, 0) = \delta(x)$ ,

$$W(x, t) = \frac{1}{\sqrt{2\pi\mathcal{F}(C \cap [0, t])}} \exp\left(\frac{-x^2}{2\mathcal{F}(C \cap [0, t])}\right). \quad (5)$$

The mean square displacement,  $\langle x^2 \rangle = 2\mathcal{F}(C \cap [0, t])$ , is proportional to  $t^\alpha$ . Hence the equation (5) gives a subdiffusive solution.

We have introduced a definition of fractal measures using fractional calculus and shown it to be useful in studying diffusion in fractal time.

One of the authors (KMK) would like to thank DST (India) (DST: PAM: GR: 381) for financial assistance.

## References

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